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On C^2 quintic spline functions over triangulations of Powell–Sabin's type

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Abstract

Given a triangulation Δ of a polygonal domain, we find a refinement Λ of Δ by choosing u_i in a neighborhood of the center of the inscribed circle of each triangle $t \in \Delta$, connecting u_i to the vertices of the triangle t , and connecting u_i to $u_{i'}$ if $t' \in \Delta$ shares an interior edge with t or to the midpoint v_e of any boundary edge e of t . The resulting triangulation is a triangulation of Powell–Sabin's type. We investigate a C^2 quintic spline space $\hat{S}_5^2(\Lambda)$ whose elements are C^3 only at u_i 's. We give a dimension formula for this spline space, show how to construct a locally supported basis, display an interpolation scheme, and prove that this spline space has the full approximation order.

Keywords: Powell–Sabin's triangulation; Bézier representation; C^2 quintic splines; Spline interpolation; Full approximation order

1. Introduction

We are interested in constructing a C^2 smooth piecewise polynomial surface over a polygonal domain Ω which interpolates or approximates a given set of scattered data over Ω . That is, for given scattered data $\{(x_i, y_i), i = 1, \dots, N\} \subset \Omega$ and for given real numbers $\{f_i^{v,\mu}, 0 \leq v + \mu \leq 2, i = 1, \dots, N\}$, construct a piecewise polynomial function $s \in C^2(\Omega)$ such that

$$D_x^v D_y^\mu s(x_i, y_i) = f_i^{v,\mu}, \quad 0 \leq v + \mu \leq 2, \quad i = 1, \dots, N.$$

Schemes for constructing C^2 interpolatory surfaces have important applications in, e.g., the design of aircraft.

There are several schemes already available in the literature. In general, there are two approaches based on the triangulation Δ of the set of scattered data: one is to construct an interpolating (or approximating) spline function of degree 8 or higher over Δ (see [6, 12]); the other is to subdivide Δ and then construct a spline function of degree as low as 5 over the refinement of Δ .

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Let

$$S_d^r(\Delta) := \{s \in C^r(\Omega): s|_t \in \mathbb{P}_d, \forall t \in \Delta\}$$

be the spline function space of smoothness r and degree d and

$$S_d^{r,\rho}(\Delta) := \{s \in S_d^r(\Delta): s \in C^\rho \text{ at each vertex of } \Delta\}.$$

be a super spline subspace with $\rho > r$, where t denotes a triangle in Δ and \mathbb{P}_d denotes the space of polynomials of total degree $\leq d$.

For examples of the second approach, Alfeld in [1] used Clough–Tocher’s refinement twice to subdivide each triangle of Δ into 9 subtriangles, and then constructed an interpolating spline function in $S_5^2(A(\Delta))$, where $A(\Delta)$ denotes Alfeld’s refinement of Δ by applying double Clough–Tocher’s refinement. Sablonnière in [13] used Powell–Sabin’s refinement method to subdivide each triangle of Δ into six subtriangles at the center of its inscribed circle and constructed interpolating spline functions in $S_5^2(\text{Sab}(\Delta))$, where $\text{Sab}(\Delta)$ denotes his variant of Powell–Sabin’s refinement of Δ . In [16], Wang subdivided each triangle of Δ into 7 subtriangles in a very special way and constructed an interpolating spline function in $S_5^2(W(\Delta))$, where $W(\Delta)$ denotes Wang’s refinement of Δ . Also, in [8], Gao subdivided each triangle of Δ into three subtriangles by using Clough–Tocher’s method and constructed an interpolating spline function in $S_6^2(\text{CT}(\Delta))$, where $\text{CT}(\Delta)$ denotes Clough–Tocher’s refinement of Δ .

Among these schemes for constructing bivariate C^2 spline interpolations over scattered data, we find that Sablonnière’s scheme uses polynomials of very lower degree and uses the least number of linear independent basis functions. (See Remark 7 in [10].) However, it requires C^3 data, i.e., $D_x^i D_y^j f$ for $i + j \leq 3$ at vertices of the given triangulation Δ and it produces an interpolant spline function which is in C^3 at all vertices of Δ and across edges of Δ . This may be a disadvantage for certain applications. Also, the approximation properties of this scheme have not been discussed in the literature.

We would like to extend Sablonnière’s study by investigating a larger C^2 quintic spline space than his spline space and investigating more triangulations than his special Powell–Sabin’s refinement of a triangulation. In addition to constructing interpolation schemes, we shall study some approximation properties of this spline space. To be more precise about what we are going to study in this paper, we need to introduce more notation and definitions.

Given a triangulation Δ of a polygonal domain, we find a refinement Δ of Δ by choosing a point \mathbf{u}_t inside triangle $t \in \Delta$ (e.g., the center of the inscribed circle of triangle t), connecting \mathbf{u}_t to the vertices of the triangle t , and connecting \mathbf{u}_t to $\mathbf{u}_{t'}$ if $t' \in \Delta$ shares an interior edge with t or to the midpoint \mathbf{v}_e of any boundary edge e of t . We choose \mathbf{u}_t (e.g., in a neighborhood of the center of triangle t) so that the line segment $\langle \mathbf{u}_t, \mathbf{u}_{t'} \rangle$ intersects at an interior point \mathbf{v}_e of the edge $e = t \cap t'$. (The original Powell–Sabin’s refinement chooses the circumcenter as \mathbf{u}_t . But it does not always yield a triangulation since \mathbf{v}_e may be outside of e . See [11].) The resulting triangulation Δ is a triangulation of Powell–Sabin’s type. Please beware of the difference between Δ and Δ . We shall investigate the following C^2 quintic spline spaces of interest:

$$\hat{S}_5^2(\Delta) = \{s \in S_5^2(\Delta): s \in C^3 \text{ at } \mathbf{u}_t, \forall t \in \Delta\}$$

and

$$\mathcal{SS}_5^2(\Delta) := \{s \in \hat{S}_5^2(\Delta): s \in C^3 \text{ at } \mathbf{v} \in \Delta\}.$$

We shall give a dimension formula for these spline spaces, show how to construct a locally supported basis, display interpolation schemes, one of which requires only C^2 data, i.e., $D_x^v D_y^\mu f(x_i, y_i)$'s with $v + \mu \leq 2$, and prove that the spline space $\mathcal{SS}_5^2(\Delta)$ has the full approximation order in the following sense: for any sufficiently smooth function f ,

$$\text{dist}(f, \mathcal{SS}_5^2) \leq C_f |\Delta|^6.$$

Here, $\text{dist}(f, \mathcal{SS}_5^2)$ denotes the distance from f to spline space \hat{S}_5^2 measured in the maximum norm on $C(\Omega)$, C_f is a constant dependent on f and the smallest angle of the triangulation. Hence, the spline space $\hat{S}_5^2(\Delta)$ has the full approximation order since $\mathcal{SS}_5^2(\Delta) \subset \hat{S}_5^2(\Delta)$.

One of the immediate applications of our study is to show the stability of the Sablonnière scheme in the sense that the scheme works well for those triangulations of Powell–Sabin's type for which \mathbf{u}_t is any point in a sufficiently small neighborhood of the center of the inscribed circle of triangle t for all $t \in \Delta$. Also, our spline space $\hat{S}_5^2(\Delta)$ may be used to model unknown target functions which are only C^2 . As we expect, the dimension of these spline spaces is larger than the number $6N$ of the interpolation conditions given in the beginning of this section. We may use the extra freedoms to adjust the quality of the interpolatory surface. For example, for a given set of scattered data, we may find an interpolant in one of these two spline spaces which minimizes the energy norm. Along these lines, the points \mathbf{u}_t may also be adjusted to yield an interpolatory surface of better quality.

It is easy to see that any triangulation of Powell–Sabin's type may be obtained from a special type of quadrangulation. For an arbitrary quadrangulation \diamond , Lai and Schumaker in [10] proposed a C^2 interpolation scheme using piecewise polynomials of degree 6 on a triangulation \oplus and showed that the scheme has approximation order 7, where the triangulation \oplus is obtained from \diamond by adding the two diagonals of each quadrangle of \diamond . Thus, the results in this paper may be viewed as a complement of [10].

The paper is organized as follows. We first introduce some notation and preliminary lemmas in Section 2 and then construct a locally supported basis, give a dimension formula for $\hat{S}_5^2(\Delta)$ and $\mathcal{SS}_5^2(\Delta)$, and prove these spline spaces have the full approximation order in Section 3.

2. Preliminaries

It is now standard to use Bézier representation to express polynomial pieces of a bivariate spline function. (We refer the interested reader to [7] or [3] for definition and properties of Bézier representation.) For any triangulation Δ , and for a spline function $s \in S_d^r(\Delta)$, s restricted to $t := \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \Delta$ may be expressed as

$$s|_t(\mathbf{x}) = \sum_{i+j+k=d} c_{ijk}^t \frac{d!}{i!j!k!} \lambda_1^i \lambda_2^j \lambda_3^k,$$

with

$$\mathbf{x} = \lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} + \lambda_3 \mathbf{w} \in t.$$

Let us define dual linear functionals $C_t^{(ijk)}$ on $S_d^0(\Delta)$ by

$$C_t^{(ijk)}(s) = c_{ijk}^t, \quad \forall s \in S_d^0(\Delta),$$

where $t = \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \Delta$ and $i + j + k = d$. Note that we have $C_t^{(d00)}(s) = s(\mathbf{u})$, $C_t^{(0d0)}(s) = s(\mathbf{v})$, and $C_t^{(00d)}(s) = s(\mathbf{w})$. We associate $C_t^{(ijk)}$ with the domain point $(i\mathbf{u} + j\mathbf{v} + k\mathbf{w})/d$ for $i + j + k = d$ and $t \in \Delta$. See Fig. 1 for all domain points of a triangulation of Powell–Sabin's type. We shall use the l th ring around \mathbf{u} to denote the collection of all coefficients $C_{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle}^{(d-l, j, k)}(s)$, $j + k = l$. We shall also use the l -disk around \mathbf{u} to denote the collection of all m th rings, $m = 0, 1, \dots, l$.

Let $A_d := (S_d^0(\Delta))^* = \{C_t^{(ijk)} : t \in \Delta, i + j + k = d\}$ be the dual space of $S_d^0(\Delta)$. Each linear functional in A_d is associated with a coefficient of any spline in $S_d^0(\Delta)$. Each linear functional in A_d is associated with a coefficient of any spline in $S_d^0(\Delta)$. For spline functions in $S_d^r(\Delta)$, the linear functionals $C_t^{(ijk)}$'s have certain relations with the linear functionals $C_{\hat{t}}^{(ijk)}$ for those \hat{t} 's that share a common interior edge with t . These relations are the well-known smoothness conditions (cf. [3, 7]). A subset \mathcal{M} of A_d is said to be a determining set for a subspace \mathcal{SS} of $S_d^r(\Delta)$ if \mathcal{M} has a property: for any $s \in \mathcal{SS}$, $\lambda(s) = 0$ for all $\lambda \in \mathcal{M}$ implies $s \equiv 0$. A subset \mathcal{M} is said to be a minimal determining set if it is a determining set and the cardinality of \mathcal{M} is minimal among all determining subsets in A_d .

In this paper, we shall give a description of a minimal determining set \mathcal{M} for spline space $\hat{S}_5^2(\Delta)$ and $\mathcal{SS}_5^2(\Delta)$. Since finding a minimal determining set around each vertex and across interior edges requires separate treatment, we need the following lemmas to deal with different cases. In Fig. 1, we show the coefficients concerned in Lemmas 1–8 and the domain points associated with the coefficients are located in the shadows.

Let $\text{star}(\mathbf{v})$ be such a triangulation that consists of all those triangles in Δ that have \mathbf{v} as one of its vertex. For each interior vertex \mathbf{v} , $\text{star}(\mathbf{v})$ has an even number of edges attached, and for each

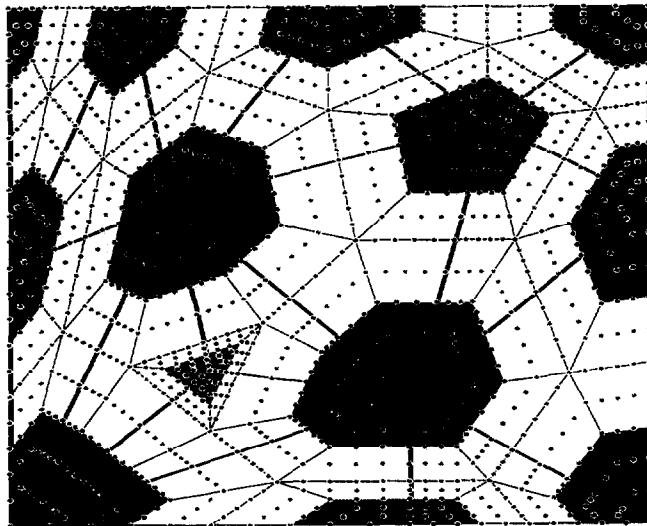


Fig. 1. Domain points of a triangulation of Powell–Sabin's type.

boundary vertex v , $\text{star}(v)$ has an odd number of edges attached since our triangulation Δ has those features. For an interior vertex v of Δ , let us denote the boundary vertices of $\text{star}(v)$ in counterclockwise direction by $v_1, w_1, v_2, w_2, \dots, v_n, w_n$. For a boundary vertex v , let the boundary vertices of $\text{star}(v)$ be $v_1, w_1, v_2, w_2, \dots, w_{n-1}, v_n$ in counterclockwise order.

Lemma 1. *Let v be a boundary vertex of Δ . The following sets form a minimal determining set for $S_3^2(\text{star}(v))$:*

- (1) $C_{\langle v, v_1, w_1 \rangle}^{(ijk)}$, $i + j + k = 3$,
- (2) $C_{\langle v, v_i, w_i \rangle}^{(030)}$, $C_{\langle v, v_i, w_i \rangle}^{(003)}$, $i = 2, \dots, n-1$,
- (3) $C_{\langle v, w_{n-1}, v_n \rangle}^{(003)}$.

These coefficients are marked with \circ in Fig. 2.

Discussion. This lemma was established in [2]. The coefficients marked with \circ in Fig. 2 can be fixed, and the rest can be found by the smoothness conditions.

Our next result deals with the case where v is an interior vertex and for some numbering of the boundary vertices of $\text{star}(v)$,

$$\angle v_1 v v_3 \leq 180^\circ. \quad (*)$$

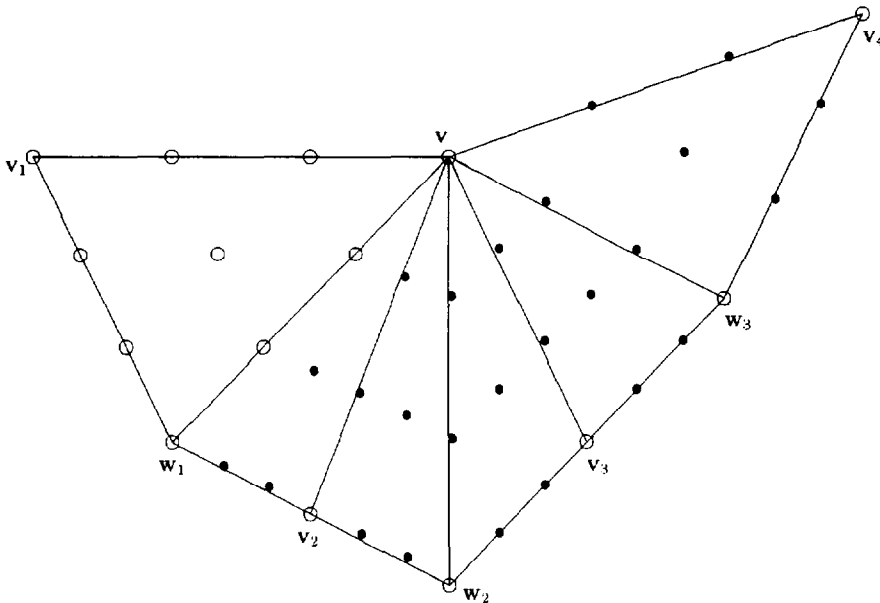


Fig. 2. A boundary vertex as in Lemma 1 ($n = 4$).

Lemma 2. Suppose that v is an interior vertex with $2n$ attached edges such that $(*)$ holds with $n \geq 4$. Then following sets form a minimal determining set for $S_3^2(\text{star}(v))$:

- (1) $C_{\langle v, w_n, v_i \rangle}^{(ijk)}$, $i + j + k = 3$,
- (2) $C_{\langle v, v_i, w_i \rangle}^{(030)}$, $i = 2, \dots, n$,
- (3) $C_{\langle v, v_i, w_i \rangle}^{(003)}$, $i = 3, \dots, n - 1$.

These coefficients are marked with \circ in the Fig. 3.

Discussion. This lemma was established in [10]. We refer the interested reader to the proof in that paper.

In our next Lemmas 3–8, we deal with the case where $n = 3$. We give a complete analysis for how to choose a minimal determining set for various situations.

Lemma 3. Suppose that v is an interior vertex of Δ with 6 edges with exactly three distinct slopes as in Fig. 4. Then the following set of 13 coefficients form a minimal determining set for $S_3^2(\text{star}(v))$:

- (1) $C_{\langle v, v_1, w_1 \rangle}^{(ijk)}$, $i + j + k = 3$,
- (2) $C_{\langle v, v_i, w_i \rangle}^{(030)}$, $i = 2, 3$,
- (3) $C_{\langle v, v_2, w_2 \rangle}^{(003)}$.

These coefficients are marked with \circ in Fig. 4.

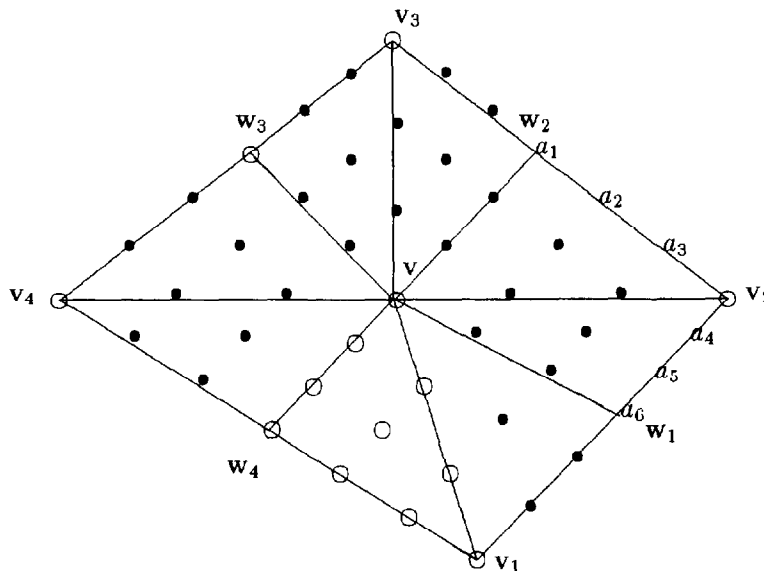


Fig. 3. An interior vertex as in Lemma 2 ($n = 4$).

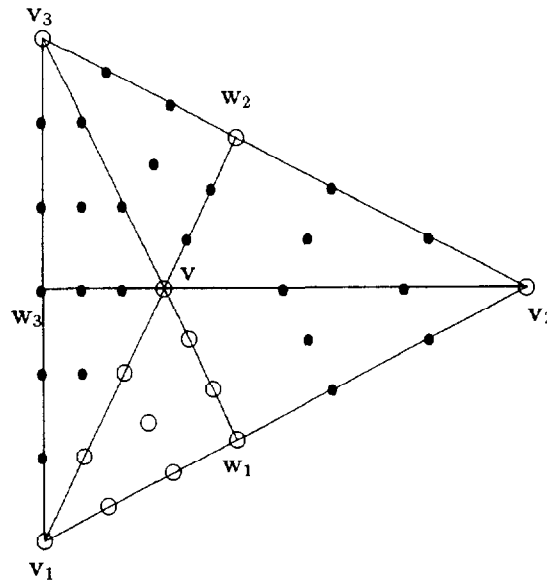


Fig. 4. An interior vertex as in Lemma 3.

Proof. By Theorem 2.1 of [14], the dimension of $S_3^2(\text{star}(v))$ is 13. To prove the lemma, we have to show that if $s \in S_3^2(\text{star}(v))$ has the above 13 coefficients equal to zero, then $s \equiv 0$. But this follows immediately from the C^1 and C^2 smoothness conditions. \square

The following Lemmas 4–6 concern the case where there are 6 edges attached to v , but there are more than 3 distinct slopes.

Lemma 4. Suppose that v has 6 interior edges with more than 3 distinct slopes. Let A_i , $i = 1, \dots, 6$ denote the area of triangles $\langle v, v_1, w_1 \rangle, \langle v, w_1, v_2 \rangle, \dots, \dots, \langle v, v_3, w_3 \rangle, \langle v, w_3, v_1 \rangle$ in counterclockwise direction around v as shown in Fig. 5. Suppose that

$$\frac{A_1 A_3 A_5}{A_2 A_4 A_6} \neq 1.$$

Then the following set of 12 coefficients form a minimal determining set for $S_3^2(\text{star}(v))$:

- (1) $C_{\langle v, v_i, w_i \rangle}^{(ijk)}$, $i + j + k = 3, i \geq 1$,
- (2) $C_{\langle v, v_i, w_i \rangle}^{(030)}$, $C_{\langle v, v_i, w_i \rangle}^{(021)}$, $i = 1, 2, 3$.

These coefficients are marked with \circ in Fig. 5.

Proof. By using Theorem 2.1 in [14], dimension of $S_3^2(\text{star}(v))$ is 12. We have 12 linear functionals. To prove this lemma, we are going to show that if there is an $s \in S_3^2(\text{star}(v))$ which is annihilated by all these 12 linear functionals, then $s \equiv 0$. Using the C^1 and C^2 smoothness conditions, it follows

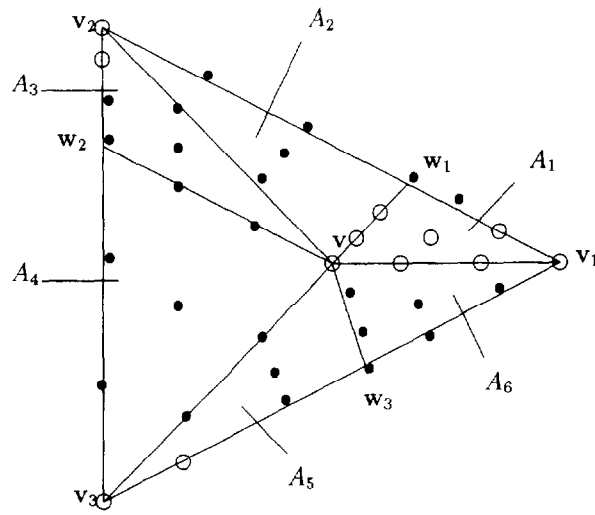


Fig. 5. An interior vertex as in Lemma 4.

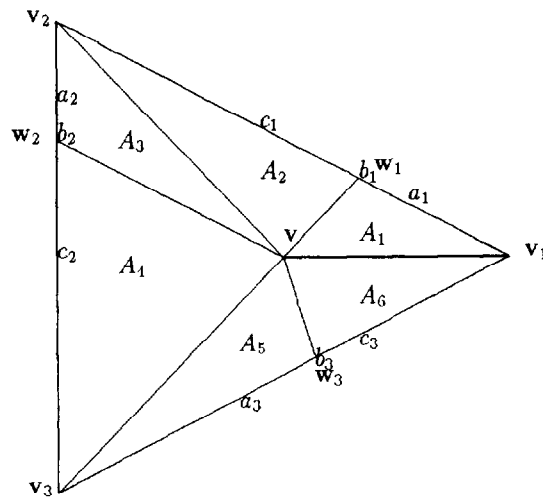


Fig. 6. An illustration for the proof of Lemma 4.

immediately that the only possible nonzero coefficients of s are those labeled a_i, b_i, c_i for $i = 1, 2, 3$ in Fig. 6. Suppose the following relations among the vertices hold:

$$w_2 = \beta_1 w_1 + \gamma_1 v_2 + \eta_1 v,$$

$$w_3 = \beta_2 w_2 + \gamma_2 v_3 + \eta_2 v,$$

$$w_1 = \beta_3 w_3 + \gamma_3 v_1 + \eta_3 v,$$

and

$$v_2 = \bar{\beta}_1 v_1 + \bar{\gamma}_1 w_1,$$

$$v_3 = \bar{\beta}_2 v_2 + \bar{\gamma}_2 w_2,$$

$$v_1 = \bar{\beta}_3 v_3 + \bar{\gamma}_3 w_3.$$

By using C^1 and C^2 smoothness conditions, we have

$$c_1 = \bar{\beta}_1 a_1 + \bar{\gamma}_1 b_1 \quad \text{and} \quad 0 = 2\bar{\beta}_1 \bar{\gamma}_1 a_1 + \bar{\gamma}_1^2 b_1.$$

Thus, $2\bar{\beta}_1 a_1 + \bar{\gamma}_1 b_1 = 0$ or $c_1 = -\bar{\beta}_1 a_1$. Similarly, we have $c_2 = -\bar{\beta}_2 a_2$ and $c_3 = -\bar{\beta}_3 a_3$.

By using C^2 smoothness condition across edges $\langle v, v_2 \rangle$, $\langle v, v_3 \rangle$, and $\langle v, v_1 \rangle$, we have

$$a_2 = \beta_1^2 c_1, \quad a_3 = \beta_2^2 c_2, \quad a_1 = \beta_3^2 c_3.$$

Thus, by substituting those equations, we have

$$\begin{aligned} a_1 &= \beta_3^2 (-\bar{\beta}_3) a_3 \\ &= \beta_3^2 (-\bar{\beta}_3) \beta_2^2 c_2 \\ &= \beta_3^2 (-\bar{\beta}_3) \beta_2^2 (-\bar{\beta}_2) a_2 \\ &= \beta_3^2 (-\bar{\beta}_3) \beta_2^2 (-\bar{\beta}_2) \beta_1^2 c_1 \\ &= \beta_3^2 (-\bar{\beta}_3) \beta_2^2 (-\bar{\beta}_2) \beta_1^2 (-\bar{\beta}_1) a_1. \end{aligned}$$

Thus, $(1 + \beta_1^2 \bar{\beta}_1 \beta_2^2 \bar{\beta}_2 \beta_3^2 \bar{\beta}_3) a_1 = 0$. Since $\beta_1 \beta_2 \beta_3 \bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3 = 1$, we have $(1 + \beta_1 \beta_2 \beta_3) a_1 = 0$. Now the condition

$$1 \neq \frac{A_1 A_3 A_5}{A_2 A_4 A_6} = -\beta_1 \beta_2 \beta_3$$

implies that $\beta_1 \beta_2 \beta_3 \neq -1$. It follows that a_1 must be zero, which in turn implies, $c_1 = 0$, $a_2 = 0$, $c_2 = 0$, $a_3 = 0$, and $c_3 = 0$ and finally, $b_1 = b_2 = b_3 = 0$. This completes the proof. \square

Lemma 5. Suppose that $\text{star}(v)$ has 6 interior edges with more than 3 distinct slopes. Let A_i , $i = 1, \dots, 6$ denote the area of triangles in counterclockwise direction around v as in Lemma 4. Let us denote by B_1, B_2, B_3 the area of triangles $\langle v, w_1, w_2 \rangle$, $\langle v, w_2, w_3 \rangle$, $\langle v, w_3, w_1 \rangle$, respectively. Suppose

$$B \neq \frac{1}{2} \left[\frac{A_4}{1 + A_5/A_6} + \frac{A_5}{1 + A_4/A_3} \right].$$

Then the following set of 12 coefficients form a minimal determining set for $S_3^2(\text{star}(v))$:

$$(1) \quad C_{\langle v, v_i, w_1 \rangle}^{(ijk)}, \quad i + j + k = 3,$$

and

$$(2) \quad C_{\langle v, v_i, w_1 \rangle}^{(030)}, \quad i = 2, 3.$$

Discussion. If the $\text{star}(\mathbf{v})$ satisfies one of the following two conditions:

$$B_1 \neq \frac{1}{2} \left[\frac{A_2}{1 + A_3/A_4} + \frac{A_3}{1 + A_2/A_1} \right]$$

and

$$B_3 \neq \frac{1}{2} \left[\frac{A_6}{1 + A_1/A_2} + \frac{A_1}{1 + A_6/A_5} \right],$$

we can find a minimal determining set for $S_3^2(\text{star}(\mathbf{v}))$ as the same as in Lemma 5.

Discussion. In particular, when B_1 or B_2 or B_3 is equal to zero, we can apply the above lemma to find a minimal determining set for $S_3^2(\text{star}(\mathbf{v}))$ since the right-hand side is always positive.

Proof of Lemma 5. As the proof of the above lemmas, we shall show that if an $s \in S_3^2(\text{star}(\mathbf{v}))$ is annihilated by all those 12 linear functionals in (1) and (2), then $s \equiv 0$. By using C^1 and C^2 smoothness conditions, many coefficients of the s are zero. In Fig. 7, we list only possible nonzero coefficients of the s .

Suppose $\mathbf{v}_3 = \beta_1 \mathbf{v}_1 + \gamma_1 \mathbf{w}_3 = \beta_2 \mathbf{v}_2 + \gamma_2 \mathbf{w}_2$, and $\mathbf{w}_3 = \alpha_3 \mathbf{v} + \beta_3 \mathbf{w}_2 + \gamma_3 \mathbf{v}_3$. Then the possible nonzero coefficients satisfy the following smoothness conditions:

$$b = \gamma_1 a, c = \gamma_1^2 a,$$

$$c = \beta_3 d, b = \beta_3^2 e + 2\gamma_3 \beta_3 d,$$

$$e = \gamma_2 f, d = \gamma_2^2 f.$$

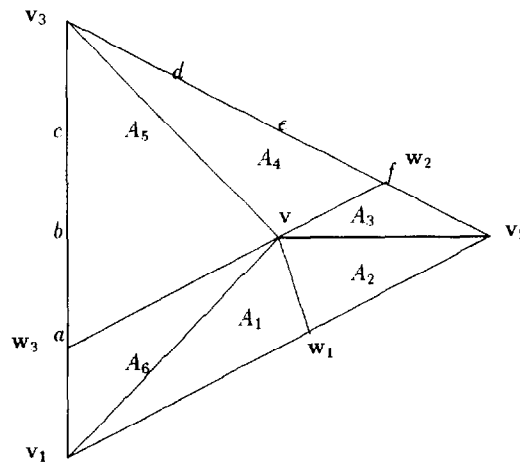


Fig. 7. An illustration for the proof of Lemma 5.

By substitutions, we have

$$\gamma_1^2 a = \beta_3 \gamma_2^2 f,$$

$$\gamma_1 a = \beta_3^2 \gamma_2 f + 2\gamma_3 \beta_3 \gamma_2^2.$$

Thus, the determinant of the above system of linear equation is

$$D := -\gamma_1 \gamma_2 \beta_3 (2\gamma_1 \gamma_2 \gamma_3 + \beta_3 \gamma_1 - \gamma_2).$$

Since

$$\beta_3 = -\frac{A_5}{A_4}, \quad \gamma_1 = \frac{A_5 + A_6}{A_6}, \quad \gamma_2 = \frac{A_4 + A_3}{A_3}, \quad \gamma_3 = \frac{B_2}{A_4},$$

we only need to show that the terms in the parenthesis of the determinant D are not equal to zero. We have

$$2\gamma_1 \gamma_2 \gamma_3 + \beta_3 \gamma_1 - \gamma_2 = \frac{2B_2(A_3 + A_4)(A_5 + A_6) - A_5(A_5 + A_6)A_3 - (A_3 + A_4)A_6 A_4}{A_3 A_4 A_6} \\ \neq 0$$

by the assumptions of this lemma. It follows that the coefficients a, b, c, d, e , and f have to be zero. This completes the proof. \square

Lemma 6. Suppose that $\text{star}(\mathbf{v})$ has 6 interior edges with more than 3 distinct slopes. Let A_1, \dots, A_6 and B_1, B_2, B_3 denote the area of triangles as in Lemmas 4 and 5. Suppose that $\text{star}(\mathbf{v})$ satisfies

$$B_2 \neq 0 \quad \text{and} \quad \frac{A_6 + A_5}{A_6} \frac{B_2}{A_4} \neq 1.$$

Then the following set form a minimal determining set for $s \in S_3^2(\text{star}(\mathbf{v}))$:

- (1) $C_{\langle \mathbf{r}, \mathbf{r}_1, \mathbf{w}_1 \rangle}^{(ijk)}, \quad i + j + k = 3,$
- (2) $C_{\langle \mathbf{r}, \mathbf{r}_2, \mathbf{w}_2 \rangle}^{(030)}, \quad \text{and} \quad C_{\langle \mathbf{r}, \mathbf{r}_2, \mathbf{w}_2 \rangle}^{(003)}.$

Proof. As in the previous proofs, we use C^1 and C^2 smoothness conditions with all annihilating linear functionals in (1) and (2), and the only possible nonzero coefficients of s are a, b, c, d as shown in Fig. 8.

Let $\mathbf{v}_3 = \alpha \mathbf{w}_3 + \beta \mathbf{v}_1$ and $\mathbf{w}_3 = \bar{\alpha} \mathbf{w}_2 + \bar{\beta} \mathbf{v}_3 + \bar{\gamma} \mathbf{v}$. Then the possible nonzero coefficients of s satisfy the following relations:

$$b = \alpha a, \quad c = \alpha^2 a, \quad c = \bar{\beta} d, \quad b = \bar{\beta}^2 d.$$

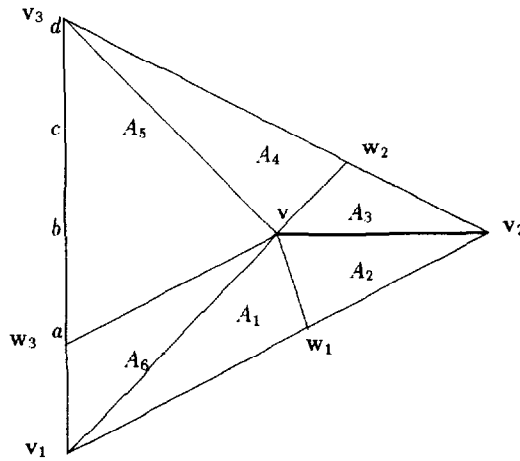


Fig. 8. An illustration for the proof of Lemma 6.

Thus, we have $\alpha a = \bar{\beta}^2 d$ and $\alpha^2 a = \bar{\beta} d$. The determinant of this system of linear equations is

$$\det \begin{bmatrix} \alpha & -\bar{\beta}^2 \\ \alpha^2 & -\bar{\beta} \end{bmatrix} = -\alpha\bar{\beta} + \alpha^2\bar{\beta}^2 = \alpha\bar{\beta}(\alpha\bar{\beta} - 1) \neq 0,$$

since

$$\alpha = \frac{A_6 + A_5}{A_6} \neq 0, \quad \bar{\beta} = \frac{B_2}{A_4} \neq 0 \quad \text{and} \quad \alpha\bar{\beta} = \frac{A_6 + A_5}{A_6} \frac{B_2}{A_4} \neq 1.$$

Therefore, the linear system has unique zero solution. This completes the proof. \square

Discussion. Similarly, we can prove that if

$$B_1 \neq 0 \quad \text{and} \quad \frac{A_4 + A_3}{A_4} \frac{B_1}{A_2} \neq 1,$$

or if

$$B_3 \neq 0 \quad \text{and} \quad \frac{A_2 + A_1}{A_2} \frac{B_3}{A_6} \neq 1,$$

we can find a minimal determining set for $S_3^2(\text{star}(v))$. Also, we can show the similar result if $[(A_4 + A_3)/A_3] (B_2/A_5) \neq 1$, or if $[(A_2 + A_1)/A_1] (B_1/A_3) \neq 1$, or if $[(A_6 + A_5)/A_5] (B_3/A_1) \neq 1$. We omit those details.

We are now going to show that if an interior vertex v with six edges attached does not satisfy one of Lemmas 4, 5, or 6, then its six edges have only three slopes.

Lemma 7. Suppose that $\text{star}(v)$ has 6 interior edges. Let A_1, \dots, A_6 denote the areas of triangles as in Lemma 4 and let B_1, B_2, B_3 denote the areas of triangles as in Lemma 5. Suppose that the triangulation satisfies the following:

$$\frac{A_1 A_3 A_5}{A_2 A_4 A_6} = 1$$

$$\frac{A_2 + A_1}{A_1} \frac{B_1}{A_3} = 1, \quad \frac{A_4 + A_3}{A_3} \frac{B_2}{A_5} = 1, \quad \frac{A_6 + A_5}{A_5} \frac{B_3}{A_1} = 1,$$

and

$$\frac{A_4 + A_3}{A_4} \frac{B_1}{A_2} = 1, \quad \frac{A_6 + A_5}{A_6} \frac{B_2}{A_4} = 1, \quad \frac{A_2 + A_1}{A_2} \frac{B_3}{A_6} = 1.$$

Then the interior edges of $\text{star}(v)$ have only three distinct slopes.

Proof. Since $A_6 A_4 = (A_1 A_3 A_5)/A_2$ and

$$B_2 \frac{A_4 + A_3}{A_3 A_5} = 1 = B_2 \frac{A_6 + A_5}{A_6 A_4},$$

we have

$$\frac{A_4 + A_3}{A_2} = \frac{A_6 + A_5}{A_1}.$$

Consider a function

$$f(w_1) = \frac{A_6 + A_5}{A_1} - \frac{A_4 + A_3}{A_2}$$

with variable w_1 along the line segment $\langle v_1, v_2 \rangle$. We know that $f(w_1)$ is monotonic from $+\infty$ to $-\infty$ when w_1 moves from v_1 to v_2 . Thus, $f(w_1) = 0$ has only one solution. The solution is when w_1, v, v_3 are colinear. That is,

$$\frac{A_6 + A_5}{A_1} = \frac{A_4 + A_3}{A_2}$$

implies that w_1, v, v_3 are colinear.

Similarly, we can show that

$$\frac{A_1 + A_2}{A_3} = \frac{A_5 + A_6}{A_4} \quad \text{and} \quad \frac{A_3 + A_4}{A_5} = \frac{A_1 + A_2}{A_6}.$$

By using the same argument as above, we can show that w_2, v, v_1 are colinear and w_3, v, v_2 are colinear. Therefore, the edges of $\text{star}(v)$ have only three distinct slopes. \square

Lemma 8. Suppose that $\text{star}(v)$ has 6 interior edges. Let A_1, \dots, A_6 denote the areas of triangles as in Lemma 4 and let B_1, B_2, B_3 denote the areas of triangles as in Lemma 5. Suppose that the triangulation satisfies the following:

$$\frac{A_1 A_3 A_5}{A_2 A_4 A_6} = 1.$$

Then any two pairs of following three pairs of equations imply the other pair of equations:

$$\frac{A_2 + A_1}{A_1} \frac{B_1}{A_3} = 1, \quad \frac{A_4 + A_3}{A_4} \frac{B_1}{A_2} = 1,$$

$$\frac{A_4 + A_3}{A_3} \frac{B_2}{A_5} = 1, \quad \frac{A_6 + A_5}{A_6} \frac{B_2}{A_4} = 1,$$

$$\frac{A_6 + A_5}{A_5} \frac{B_3}{A_1} = 1, \quad \frac{A_2 + A_1}{A_2} \frac{B_3}{A_6} = 1.$$

Proof. Let us suppose that the first and third pair of equations hold. By using the same argument as in the proof of Lemma 7, we have w_2, v, v_1 are colinear and w_3, v, v_2 are collinear. Thus, we have

$$\frac{B_3}{A_2} = \frac{B_2}{A_3} \quad \text{and} \quad \frac{B_1}{A_1} = \frac{B_2}{A_6}.$$

Since

$$A_2 A_4 = \frac{A_1 A_3 A_5}{A_6} \quad \text{and} \quad A_1 A_5 = \frac{A_2 A_4 A_6}{A_3},$$

we have

$$\begin{aligned} 1 &= \frac{A_4 + A_3}{A_4} \frac{B_1}{A_2} = \frac{A_4 + A_3}{A_1 A_3 A_5} A_6 B_1 \\ &= \frac{A_4 + A_3}{A_1 A_3 A_5} A_1 B_2 \\ &= \frac{A_4 + A_3}{A_3} \frac{B_2}{A_5} \end{aligned}$$

and

$$\begin{aligned} 1 &= \frac{A_6 + A_5}{A_5} \frac{B_3}{A_1} = \frac{A_6 + A_5}{A_2 A_4 A_6} A_3 B_3 \\ &= \frac{A_6 + A_5}{A_2 A_4 A_6} A_2 B_2 \\ &= \frac{A_6 + A_5}{A_6} \frac{B_2}{A_4}. \end{aligned}$$

Hence, we are able to apply Lemma 6 to conclude that $\text{star}(v)$ has only three distinct slopes of interior edges attached to v . \square

So far we have studied the smoothness conditions around each interior vertices of Δ . Recall v_e is the vertex of Δ on an edge e of Δ . To consider the smoothness conditions around v_e for each interior edge e , we need the following lemma.

Lemma 9. *Let q be a convex quadrangle with vertices w_1, w_2, w_3 , and w_4 and Δ_q be the triangulation of q by using two diagonals of q . Denote the intersection of the two diagonals by v_e . Consider $S_3^0(\Delta_q)$. The following set form a determining set for $S_3^2(\Delta_q)$:*

$$(1) \ C_{\langle v_e, w_l, w_{l+1} \rangle}^{(ijk)}, \quad j \geq 2, l = 1, 2, 3, 4,$$

$$(2) \ C_{\langle v_e, w_l, w_{l+1} \rangle}^{(ijk)}, \quad k \geq 2, l = 1, 2, 3, 4$$

and

$$(3) \ C_{\langle v_e, w_1, w_2 \rangle}^{(500)}.$$

Proof. For any $s \in S_3^2(\Delta_q)$, annihilated by those linear functionals, the only possible nonzero coefficients of s are $c_{\langle v_e, w_l, w_{l+1} \rangle}^{(ijk)}$, $i = 1, 2, 3, 4$ and $c_{\langle v_e, w_l, w_{l+1} \rangle}^{(ijk)}$, $i = 1, 2, 3, 4$.

By using Lemma 4.1 in [4], we can see that $c_{\langle v_e, w_l, w_{l+1} \rangle}^{(ijk)} = 0$, $l = 1, 2, 3, 4$. By using Lemma 4.1 in [4] again, we now see that the remaining 4 coefficients $c_{\langle v_e, w_l, w_{l+1} \rangle}^{(ijk)}$, $i = 1, 2, 3, 4$, have to be zero. This completes the proof. \square

Finally, we give a lemma regarding a minimal determining set for $S_3^3(\text{star}(v))$ for $v = u_t$ for any triangle $t \in \Delta$. In Fig. 9, we show the coefficients concerned in the following lemma and the domain points associated with these coefficients are located in the shadowy area around each u_t .

Lemma 10. *Let $v = u_t$ be an interior vertex of Δ . Then the following set of 10 coefficients form a minimal determining set for $S_3^3(\text{star}(v))$:*

$$(1) \ C_{\langle v, v_i, v_j \rangle}^{(030)}, \quad C_{\langle v, v_i, v_j \rangle}^{(021)}, \quad C_{\langle v, v_i, v_j \rangle}^{(120)}, \quad i = 1, 2, 3,$$

$$(2) \ C_{\langle v, v_1, w_1 \rangle}^{(102)}.$$

These coefficients are marked with \circ in Fig. 10.

Proof. Clearly, the space $S_3^3(\text{star}(v))$ is just the space of polynomials of total degree 3. Thus, it has dimension 10. For any spline function $s \in S_3^3(\text{star}(v))$, by setting the those 10 linear functionals to be zero, we see that coefficients $c_{\langle v, w_i, v_{i+1} \rangle}^{(ijk)} = 0$, $i = 1, 2, 3$ with $v_4 = v_1$. Using Lemma 4.1 in [4], it follows that $c_{\langle v, v_i, w_i \rangle}^{(ijk)} = 0$ and $c_{\langle v, v_i, w_i \rangle}^{(ijk)} = 0$ for $i = 1, 2, 3$. Since $c_{\langle v, v_1, w_1 \rangle}^{(ijk)} = 0$, we use Lemma 4.1 again to conclude $c_{\langle v, v_1, w_1 \rangle}^{(ijk)} = 0$ and $c_{\langle v, w_1, v_2 \rangle}^{(ijk)} = 0$. The C^2 smoothness condition implies $c_{\langle v, v_2, w_2 \rangle}^{(ijk)}$ and $c_{\langle v, v_2, w_2 \rangle}^{(ijk)}$. Similarly, the C^2 smoothness condition implies $c_{\langle v, w_3, v_1 \rangle}^{(ijk)} = 0$ and $c_{\langle v, w_3, v_1 \rangle}^{(ijk)} = 0$. Using Lemma 4.1 in [4], we get $c_{\langle v, v_2, w_2 \rangle}^{(ijk)} = 0 = c_{\langle v, w_2, v_3 \rangle}^{(ijk)} = 0$ and $c_{\langle v, w_3, v_1 \rangle}^{(ijk)} = 0 = c_{\langle v, v_3, w_3 \rangle}^{(ijk)}$. We now use C^1 smoothness condition to conclude that all remaining coefficients have to be zero. This completes the proof. \square

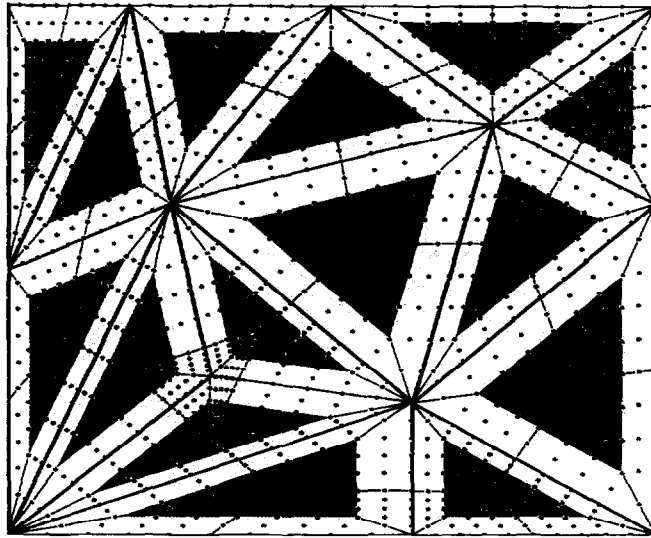


Fig. 9. The coefficients concerned in Lemma 10.

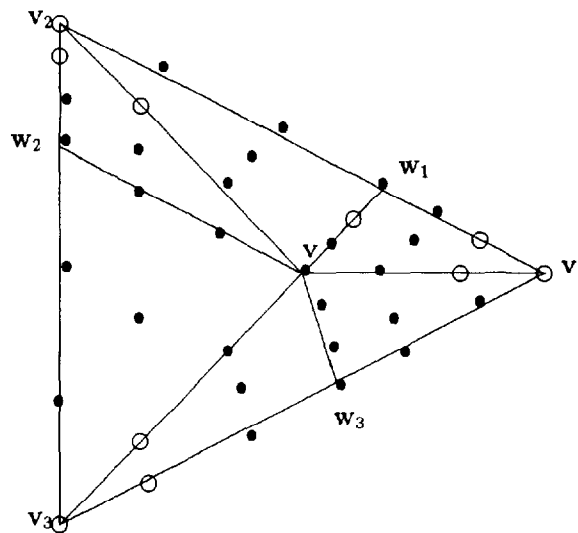


Fig. 10. An interior vertex as in Lemma 10.

Discussion. In Lemma 10, our condition (2) may be replaced by an interpolation condition at v , i.e., (2') $C_{\langle v, v_1, w_1 \rangle}^{(3,0,0)}$. We use (2) instead of (2') for a convenient proof of this lemma.

3. Dimension, locally supported basis, interpolation schemes, and approximation power of $\hat{S}_5^2(\Delta)$

Let \mathcal{V}_I and \mathcal{V}_B be the collections of all interior and boundary vertices of Δ . Let \mathcal{E}_I and \mathcal{E}_B be the collections of all interior and boundary edges of Δ . Set $\mathcal{U} := \{u_t : t \in \Delta\}$ and $\mathcal{V}_E := \{v_e : e \in \mathcal{E}_I \cup \mathcal{E}_B\}$.

Recall our two spline subspaces are as follows:

$$\hat{S}_3^2(\Delta) := \{s \in S_3^2(\Delta): s \in C^3 \text{ at } v \in \mathcal{U}\}$$

and

$$\mathcal{SS}_3^2(\Delta) := \{s \in \hat{S}_3^2(\Delta): s \in C^3 \text{ at } v \in \mathcal{V}_1 \cup \mathcal{V}_B\}.$$

Theorem 1. *The dimension formulas of those spline spaces are*

$$\dim \hat{S}_3^2(\Delta) = 6(V_1 + V_B) + 4T + 3E + V_B + V_s$$

$$\dim \mathcal{SS}_3^2(\Delta) = 10(V_1 + V_B) + T + E + V_B$$

where V_1, V_B, E denote the cardinality of sets $\mathcal{V}_1, \mathcal{V}_B, \mathcal{V}_E$, T be the number of triangles of Δ , and V_s denotes the number of vertices of Δ satisfying Lemma 3.

Proof. We shall first construct a minimal determining set of \mathcal{A}_5 for $\hat{S}_3^2(\Delta)$ and for $\mathcal{SS}_3^2(\Delta)$. In order to do that, we need to use following notations:

$\mathcal{V}_3 := \{v \in \mathcal{V}_1: \text{star}(v) \text{ satisfies the conditions in Lemma 3}\};$

$\mathcal{V}_4 := \{v \in \mathcal{V}_1: \text{star}(v) \text{ satisfies the conditions in Lemma 4}\};$

$\mathcal{V}_5 := \{v \in \mathcal{V}_1: \text{star}(v) \text{ satisfies the conditions in Lemma 5}\};$

$\mathcal{V}_6 := \{v \in \mathcal{V}_1: \text{star}(v) \text{ satisfies the conditions in Lemma 6}\}.$

We note that for an interior vertex v of Δ with exactly three edges attached, if v is not in $\mathcal{V}_4 \cup \mathcal{V}_5 \cup \mathcal{V}_6$, then v is in \mathcal{V}_3 by Lemma 7.

Algorithms for constructing a minimal determining set for $\hat{S}_3^2(\Delta)$ and \mathcal{SS}_3^2 are given as follows.

Algorithm 3.1. Choose the linear functionals associated with the following coefficients:

- (1) 6 coefficients determining the 2-disk around each vertex v of Δ ;
- (2) for each $v \in \mathcal{V}_6 \setminus (\mathcal{V}_3 \cup \mathcal{V}_4 \cup \mathcal{V}_5)$, enough coefficients on the 3rd ring around v to determine $S_3^2(\text{star}(v))$ as in Lemma 6;
- (3) for remaining vertex $v \in \mathcal{V}_1 \cup \mathcal{V}_B$, enough additional coefficients on the 3rd ring around v to determine $S_3^2(\text{star}(v))$ as in Lemmas 1–5;
- (4) one coefficient of the form c_{203} as in item (2) of Lemma 10 for each interior vertex $v \in \mathcal{U}$;
- (5) one coefficient at $v_e \in \mathcal{V}_E$ and one additional coefficient in the 1-disk around each v_e for boundary edge e of Δ .

Algorithm 3.2. Choose the linear functionals associated with the following coefficients:

- (1) 10 coefficients determining the 3-disk around each vertex v of Δ ;
- (2) one coefficient of the form c_{203} as in item (2) of Lemma 10 for each interior vertex $v \in \mathcal{U}$;
- (3) one coefficient at $v_e \in \mathcal{V}_E$ and one additional coefficient in the 1-disk around each v_e for boundary edge e of Δ .

Let \mathcal{C}_1 and \mathcal{C}_2 be the collections of the linear functionals in Algorithms 3.1 and 3.2, respectively. To show that \mathcal{C}_1 is a determining set, we use the C^1 and C^2 smoothness conditions across all interior edges along with Lemmas 1–10. Suppose we set all of the coefficients of $s \in \hat{S}_5^2(\Delta)$ in \mathcal{C}_1 to zero. Then by (1)–(3) of Algorithm 3.1 and by Lemmas 1–6, all coefficients in the 3-disks around every vertex of $\mathcal{T}_1 \cup \mathcal{T}_B$ must be zero. Since the coefficient in (4) are zero, we see, by Lemma 10, all coefficients in the 3-disk around each vertex of \mathcal{U} must be zero. Finally, by Lemma 9, the coefficients of s around every vertex v_e of \mathcal{T}_E are zero. Thus, s must vanish identically over Ω . This completes the proof that \mathcal{C}_1 is a determining set. Similarly, we can show that \mathcal{C}_2 is also a determining set by using C^3 smoothness conditions and Lemmas 9 and 10 only.

Let m_1 and m_2 be the cardinality of \mathcal{C}_1 and \mathcal{C}_2 , respectively. Then the above argument shows that $\dim(\hat{S}_5^2(\Delta)) \leq m_1$ and $\dim(\mathcal{S}\mathcal{S}_5^2(\Delta)) \leq m_2$.

Next we construct a locally supported basis for $\hat{S}_5^2(\Delta)$ and $\mathcal{S}\mathcal{S}_5^2(\Delta)$. Let us number the linear functionals in \mathcal{C}_1 by $\gamma_i, i = 1, \dots, m_1$. We now show that for each such i , there exists a spline $B_i \in \hat{S}_5^2(\Delta)$ such that

$$\gamma_j B_i = \delta_{ij} \quad \text{for all } i, j = 1, \dots, m_1.$$

To construct B_i we describe how to choose its coefficients. First, we set $\gamma_i(B_i) = 1$ and $\gamma_j(B_i) = 0$ for all other $1 \leq j \leq m_1$. Then we solve for the remaining coefficients using Lemmas 1–10. Clearly, the splines B_i are linearly independent. We conclude that $\dim(\hat{S}_5^2(\Delta)) \geq m_1$ and hence, $\dim(\hat{S}_5^2(\Delta)) = m_1$ and $\{B_i, i = 1, \dots, m_1\}$ form a basis for $\hat{S}_5^2(\Delta)$.

To describe the support properties of the basis spline B_i , we note the association between coefficients and linear functionals. Suppose c_i is the coefficient of B_i which is set to 1. If c_i is within the 3-disk around some vertex $v \in \mathcal{T}_1 \cup \mathcal{T}_B$, the support of B_i is all triangle of Δ sharing v . These kinds of basis splines can arise when choosing linear functionals as in item (1)–(3) of Algorithm 3.1. If c_i is within the 2-disk around some vertex $u_t \in \mathcal{U}$, the support of B_i is the triangle t . This type of basis splines can arise when choosing linear functionals as in item (5) of Algorithm 3.1. If c_i is within the 1-disk around some vertex $v_e, e \in \mathcal{E}_1$, then the support of B_i is the quadrilateral consisting of two vertices $u_t, u_{t'}$ and two vertices of e where $t \cap t' = e$. If c_i is within the 1-disk around some vertex $v_e, e \in \mathcal{E}_B$, then the support of B_i is a triangle consisting of a vertex u_t and two vertices of e with $e \in t$. These two kinds of basis splines can arise from choosing linear functionals as in item (6) of Algorithm 3.1.

Similarly, we can construct a locally supported basis for $\mathcal{S}\mathcal{S}_5^2(\Delta)$ and prove its dimension is m_2 . We omit those details here.

With those locally supported basis splines B_i 's and linear functionals $\mathcal{C}_1 = \{\gamma_i, i = 1, \dots, m_1\}$, we are able to display an interpolation scheme based on $\hat{S}_5^2(\Delta)$. Also, letting $\{\tilde{B}_i, i = 1, \dots, m_2\}$ be the locally supported basis for spline space $\mathcal{S}\mathcal{S}_5^2(\Delta)$ and writing $\mathcal{C}_2 = \{\tilde{\gamma}_i, i = 1, \dots, m_2\}$, we can construct another interpolation scheme based on $\mathcal{S}\mathcal{S}_5^2(\Delta)$.

Theorem 2. For any prescribed set of real numbers $\{f_i\}_{i=1}^{m_1}$, there exists a unique spline $S_f \in \hat{S}_5^2$ such that

$$\gamma_i(S_f) = f_i, \quad i = 1, \dots, m_1$$

and S_f is given below

$$S_f = \sum_{\gamma_i \in \mathcal{C}_1}^m f_i B_i.$$

Also, for any prescribed set of real numbers $\{f_i\}_{i=1}^{m_2}$, there exists a unique spline $L_f \in \mathcal{S}\mathcal{S}_5^2$ such that

$$\hat{\gamma}_i(L_f) = f_i, \quad i = 1, \dots, m_2$$

and L_f is given below

$$L_f = \sum_{\tilde{\gamma}_i \in \mathcal{C}_2}^m f_i \tilde{B}_i.$$

Note that setting the 6 coefficients in a 2-disk around a vertex is equivalent to setting the function value, gradient, and Hessian at that vertex. Thus, these two schemes solve the scattered data interpolation problem mentioned in the beginning of Section 1.

To establish the approximation order of spline space $\hat{S}_5^2(\Delta)$, we actually study that of the spline space $\mathcal{S}\mathcal{S}_5^2(\Delta)$ using interpolation scheme L_f since $\text{dist}(f, \hat{S}_5^2(\Delta)) \leq \text{dist}(f, \mathcal{S}\mathcal{S}_5^2(\Delta))$.

Theorem 3. For any given triangulation Δ , let Δ be a Powell–Sabin’s refinement of Δ . Then the approximation order of $\hat{S}_5^2(\Delta)$ is full.

Proof. We use the linear operator L_f defined in Theorem 2 above. Clearly, $L_p = p$ for all $p \in \mathbb{P}_5$ since $p \in \mathcal{S}\mathcal{S}_5^2(\Delta)$. Note that \tilde{B}_i ’s are locally supported basis. We claim that there is a constant K depending only on the smallest angle in the triangulation Δ such that each basis function \tilde{B}_i is bounded by K . Indeed, fix $1 \leq i \leq m_2$. Since on each triangle t , \tilde{B}_i is just a linear combination of the Bernstein basis functions associated with that triangle, to establish a bound on \tilde{B}_i it is enough to bound its coefficients. Clearly, the coefficients of \tilde{B}_i which were set in the construction are bounded by 1. The remaining coefficients of \tilde{B}_i are computed from C^1 , C^2 and C^3 smoothness conditions or by using Lemma 4.1 in [4] and C^2 smoothness condition as in Lemma 10. Thus, the coefficients are bounded by a constant depending on

$$\eta(\Delta) = \max \left\{ \frac{\text{area}(t)}{\text{area}(t')}: t, t' \text{ are neighbouring triangles in } \Delta \right\}.$$

Since this quantity depends on the ratios of the area of neighboring triangles, it can be bounded by a constant depending only on the smallest angle in Δ .

By Stein’s extension theorem [15, p. 181], we may assume that f is defined on the smallest disk $\hat{\Omega}$ containing Ω with $\|D^l f\|_{\hat{\Omega}} \leq K \|D^l f\|_{\Omega}$ for $l = 0, \dots, 6$. Here, $\|D^l f\|_{\Omega} := \max \{ |D_x^i D_y^j f(x, y)|, i + j \leq l, (x, y) \in \Omega \}$ denotes the maximum norm of $f \in C^l(\Omega)$.

We need to use a result in [5]: if a linear functional F defined on $C^{k+1}(H)$ satisfies the following two properties

- (1) $|F(f)| \leq C \sum_{l=0}^k h^l \|D^l f\|_H$, where C is independent of f and h and
- (2) $F(p) = 0$ for all $p \in \mathbb{P}_k$, where H is a disk and h is its diameter

then there exists a positive constant K independent of f and h such that

$$|F(f)| \leq Kh^{k+1} \|D^{k+1} f\|_H.$$

Fix a point $(x, y) \in \Omega$. Let t be the triangle containing (x, y) . We denote the union of the supports of all B_i 's such that $B_i|_t \neq 0$ by H_t and let H be the smallest disk containing H_t . Then H has the size $h \leq 3|\Delta|$ with $|\Delta|$ being the maximum of the diameter of all triangles of Δ .

Consider a linear functional F defined by $F(f) = f(x, y) - L_f(x, y)$. It is easy to see that our linear functional F satisfies the above two properties. Indeed, F satisfies (2) for $k = 5$ as mentioned above.

Next we recall that the coefficients of each spline s on a triangle t are directly related to values of s and its derivatives at the vertices of t . Write $t = \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$. In particular, $C_t^{500}(s) = s(\mathbf{u})$, $C_t^{050}(s) = s(\mathbf{v})$, and $C_t^{005}(s) = s(\mathbf{w})$, and in general (cf. [9]), for all $i + j + k = 5$,

$$C_t^{(ijk)}(s) = \sum_{\substack{v_1 \leq j \\ v_2 \leq k}} \binom{j}{v_1} \binom{k}{v_2} \frac{(5 - v_1 - v_2)!}{5!} D_{\mathbf{v}-\mathbf{u}}^{v_1} D_{\mathbf{w}-\mathbf{u}}^{v_2} s(\mathbf{u}),$$

where in general, $D_{\mathbf{a}-\mathbf{b}}$ is the directional derivative

$$D_{\mathbf{a}-\mathbf{b}} f(x) = \lim_{h \rightarrow 0} \frac{f(x + h(\mathbf{a} - \mathbf{b})) - f(x)}{h}$$

in the direction $\mathbf{a} - \mathbf{b}$. Directly estimating from the expression of linear functionals above, we conclude that our F satisfies (1) for $k = 5$ with $h = |\Delta|$ with the constant K dependent on the smallest angle of Δ .

Thus, we immediately obtain that

$$|f(x, y) - L_f(x, y)| \leq Kh^6 \|D^6 f\|_H \leq Kh^6 \|D^6 f\|_\Omega.$$

Here the constant K changes from step to step. This is true for any $(x, y) \in \Omega$. Thus, we have established the result of this theorem. \square

We end this paper with the following remark. For a given triangulation of Powell–Sabin's type, if a vertex of the triangulation satisfies the condition in Lemma 4 or Lemma 5 or Lemma 6 marginally, we may require that the approximating spline function to be constructed should be C^3 at this vertex.

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